thermodynamics of effective stress in partially saturated soils

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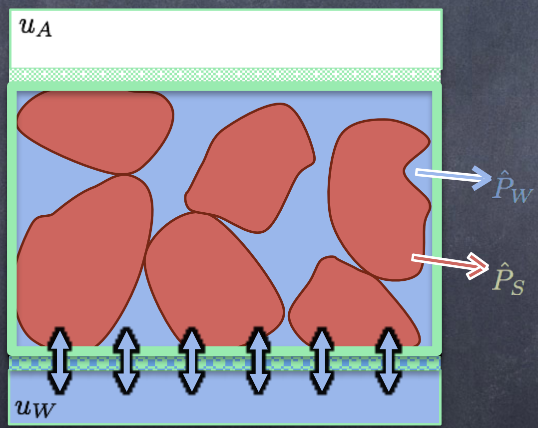
Terzaghi's effective stress

$$\sigma_{ij}^{eff} = \sigma_{ij} - P_T \delta_{ij}$$

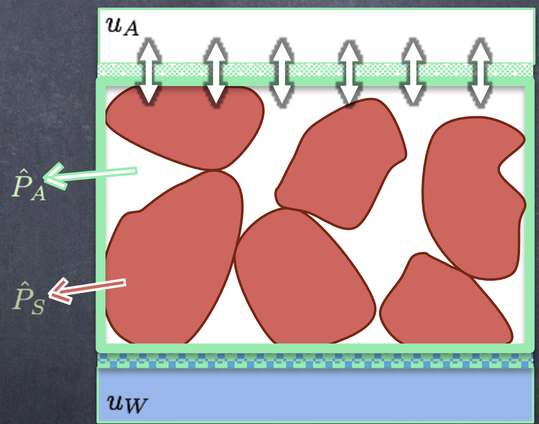
Sr=1, fully wet



$$u_W = \hat{P}_W = \hat{P}_S = u_A$$
$$s = u_A - u_W = 0$$

$$P_T = u_W$$

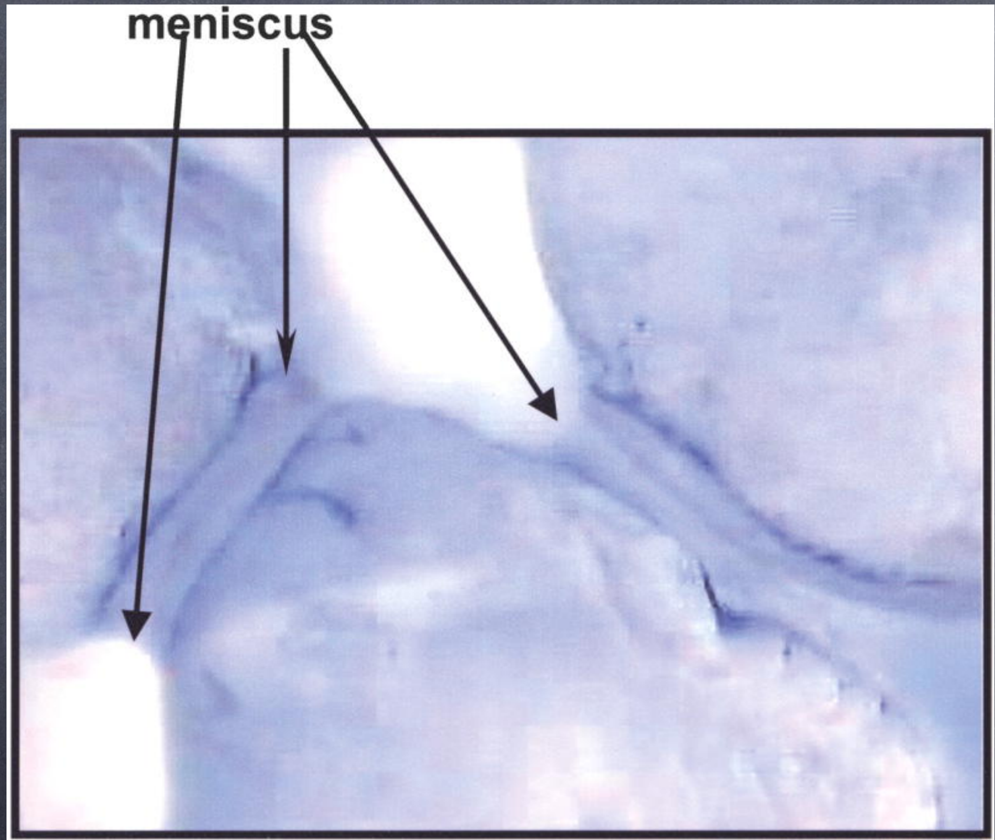
Sr=0, fully dry



$$u_A = \hat{P}_A = \hat{P}_S = u_W$$
$$s = u_A - u_W = 0$$

$$P_T = u_A$$

partially saturated media



Solid-Air-Water interfaces

Crist et al., 2004

thermodynamic treatments

- previous treatments try to follow microscopic surface tensions of complex shaped interfaces
- current treatment follows macroscopic suction from soil-water retention curves

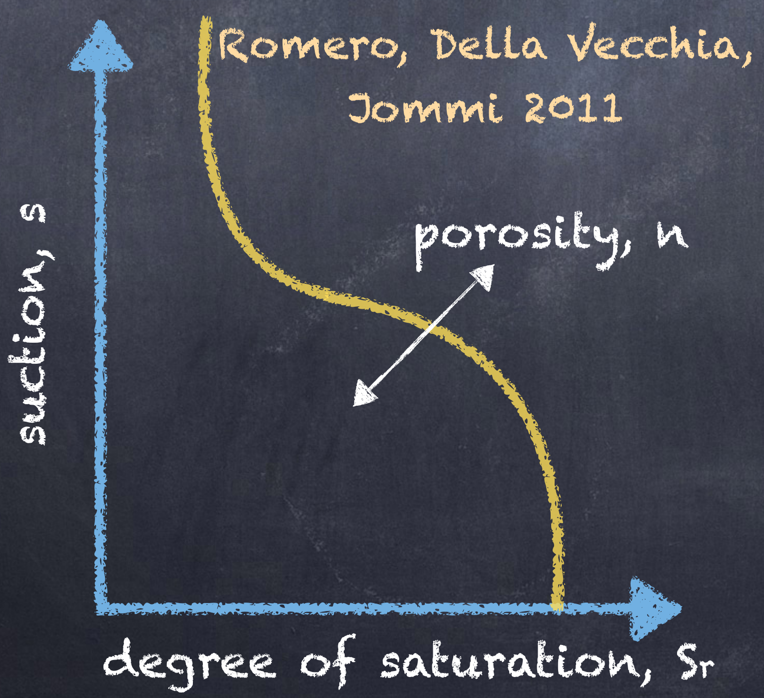
Note, soil-water retention curves encode the effects of surface tensions of complex interfaces

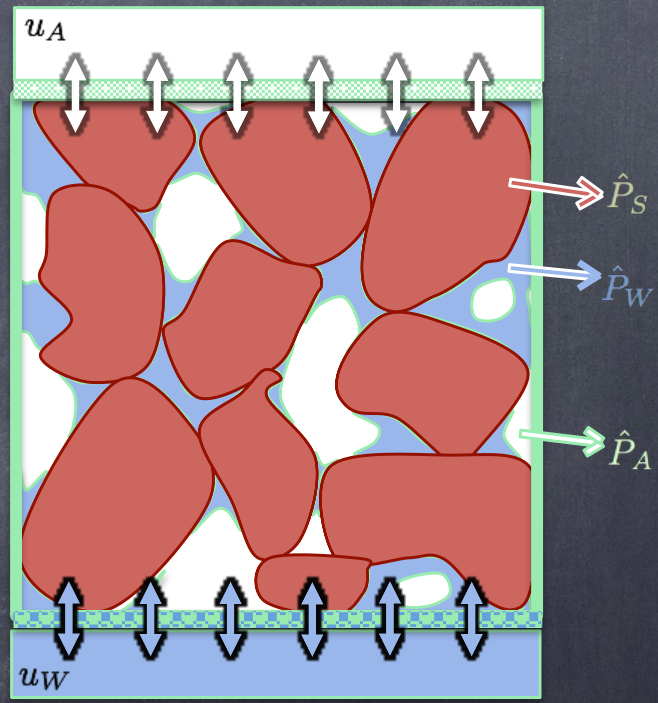
partially saturated media ($1>S_r>0$)

$$\phi_\beta \equiv \frac{V_\beta}{V}, \quad \sum_\beta \phi_\beta = 1, \quad \beta = (A, W, S)\,,$$

$$n \equiv \phi_W + \phi_A, \quad S_r \equiv \frac{\phi_W}{n}.$$

soil-water retention curves





$$s = u_A - u_W$$

$$\stackrel{?}{=} \hat{P}_A - \hat{P}_W$$

Bishop's effective stress

$$\sigma_{ij}^{eff} = \sigma_{ij} - P_T \delta_{ij}$$

partially saturated soils (1>Sr>0)

$$P_T = u_A - \chi \left(u_A - u_W \right) \quad = s$$

$$\chi \equiv \chi \left(S_r \right) \quad \ldots\text{or}\ldots \quad \chi \equiv \chi \left(s \right) \quad \ldots\text{or???}$$

but soil-water retention shows s=s(Sr,n)

methodology

- develop a hydrodynamic treatment without needing to assume the shapes of interfaces
- uncover the meaning of pressure P_T
- distinguish intrinsic and partial quantities
- reveal the structure of effective stress

partial quantities

$\varrho_\beta \equiv M_\beta/V, \ \beta = (A, W, S)$ =partial densities

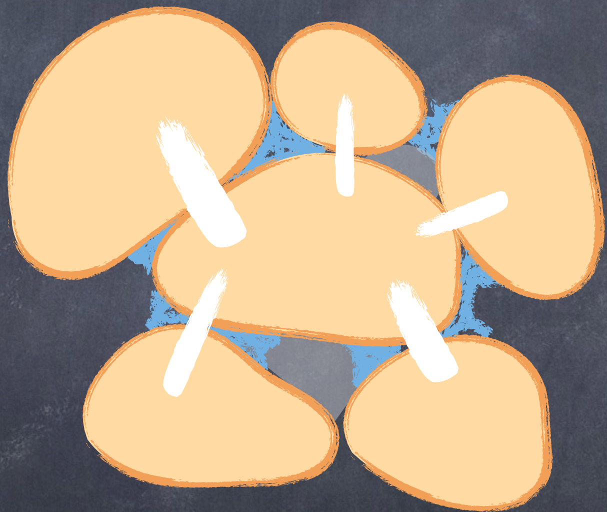
$\varrho \equiv \sum \varrho_\beta$ =total density

$f \equiv f(\varrho_\beta) = f(\varrho_A, \varrho_W, \varrho_S)$ =free energy

$\mu \equiv \partial f/\partial \varrho$ =chemical potential (Gibbs, 1878)

general hydrodynamic result

$$\sigma^e_{ij} = \sigma_{ij} - P_T \delta_{ij}$$



$$P_T = -\frac{\partial(f/\varrho)}{\partial(1/\varrho)} = \boxed{P_T \equiv \varrho\mu - f}$$ =thermodynamic pressure

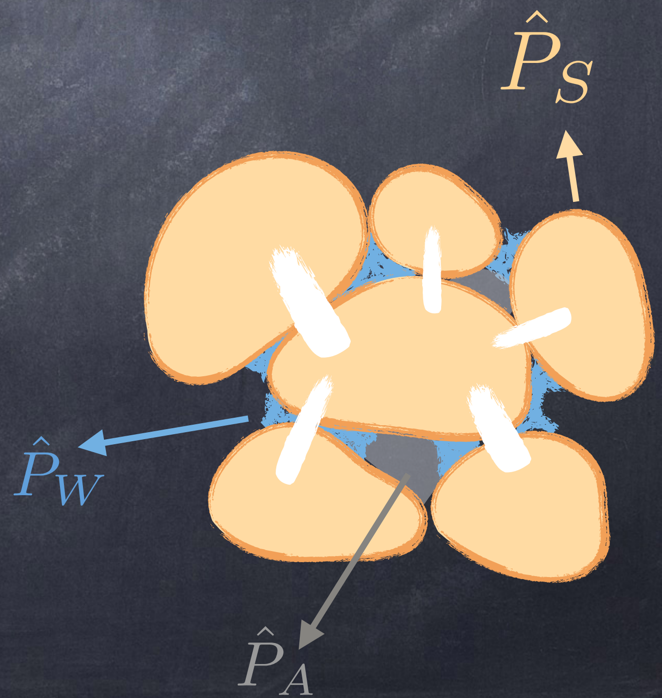
intrinsic quantities

$\hat{f}_\beta \equiv \hat{f}_\beta(\hat{\varrho}_\beta),$ =intrinsic free

$\hat{\mu}_\beta \equiv \partial \hat{f}_\beta / \partial \hat{\varrho}_\beta,$ =intrinsic chemical

$\hat{P}_\beta \equiv \hat{\varrho}_\beta \hat{\mu}_\beta - \hat{f}_\beta.$ =intrinsic pressure

$f = \sum \phi_\beta \hat{f}_\beta(\hat{\varrho}_\beta)$ =total free

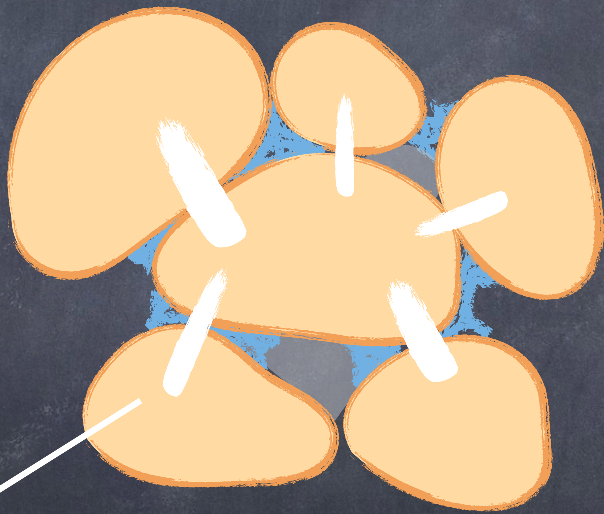


(total) thermodynamic pressure

$$P_T = \sum_\beta \hat{P}_\beta \left[1 - \rho \frac{\partial}{\partial \rho}\right] \phi_\beta$$

effective stress and elastic forces

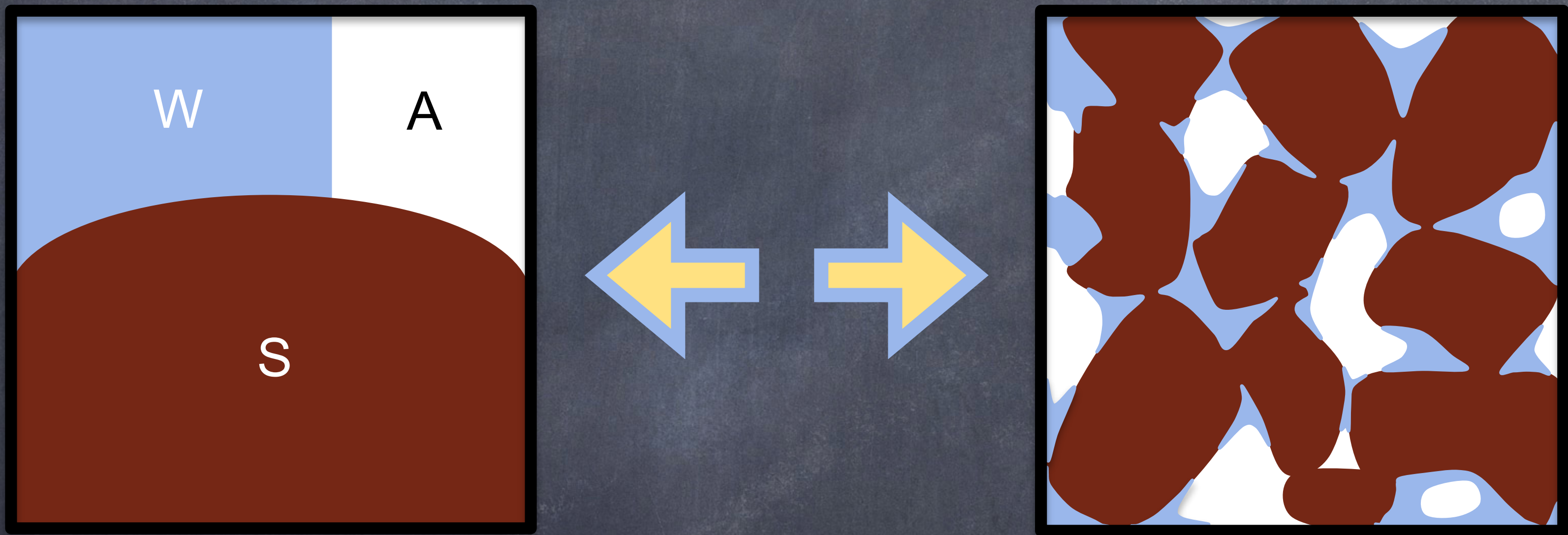
$$\sigma_{ij}^{eff} = \sigma_{ij} - P_T \delta_{ij}$$



$$\sigma_{ij}^{eff} = \sigma_{ij}^{e}$$

(elastic) contact forces → $P^{eff} = \frac{1}{3}\sigma_{ii}^{eff}$ → use to integrate constitutive relations

‘suctionless limit’ and ‘common pressure’



$$P_T = P_0 = \hat{P}_\beta$$

$$\Delta \hat{P}_\beta = \hat{P}_\beta - P_0 = K_\beta \frac{\Delta \hat{\varrho}_\beta}{\hat{\varrho}_\beta}$$

minimisation of free energy

$$\delta \int \left(f - L_1 \sum_\beta \phi_\beta - L_2 [\hat{P}_A - \hat{P}_W] \right) d^3 r = 0$$

mass constraint

$= \hat{s}$

suction constraint

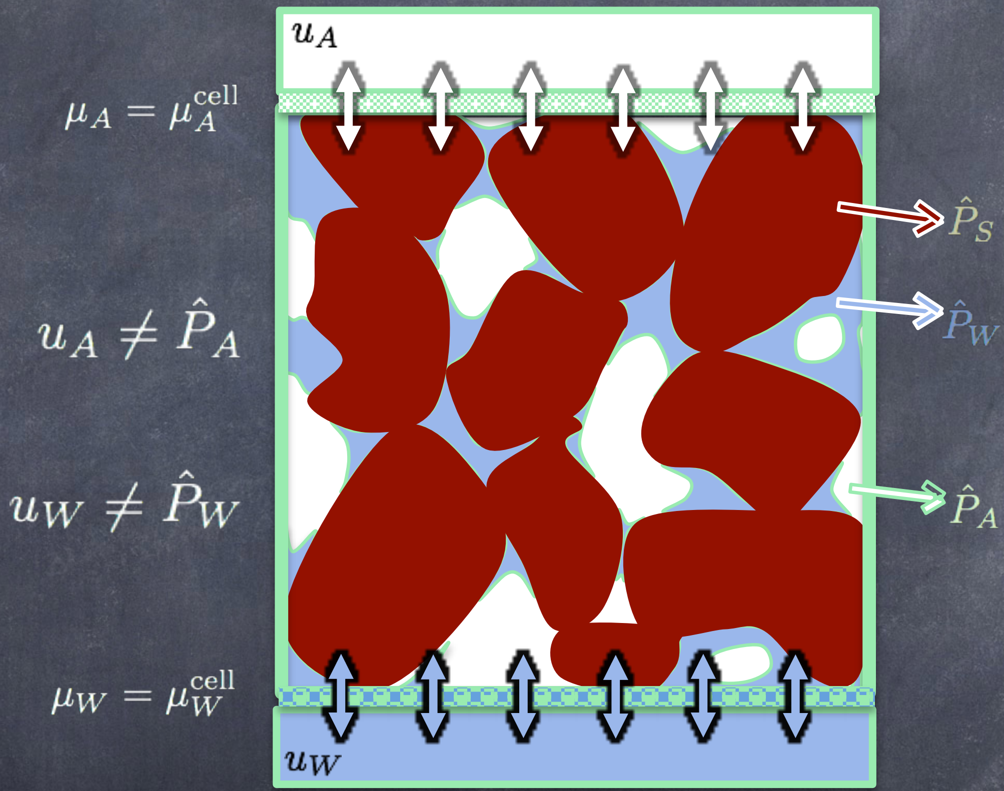
$$\hat{s} = L_2 \left(\frac{K_A}{\phi_A} + \frac{K_W}{\phi_W} \right) \quad \text{where} \quad K_\beta = \hat{\varrho}_\beta \frac{\partial \hat{P}_\beta}{\partial \hat{\varrho}_\beta}$$

...and by accounting for realistic compressibilities:

$$\hat{s} = \hat{P}_A - \hat{P}_W$$

$$= \Delta \hat{P}_A - \Delta \hat{P}_W$$

equilibrium and suctions



$$\mu_\beta = \mu_\beta^{\text{cell}}$$

$s = u_A - u_W$ ='measured suction'

$\hat{s} = \hat{P}_A - \hat{P}_W$ ='intrinsic suction'

$$\hat{s} \neq s$$

second order approximation around suctionless limit

$$\Delta u_\beta = \frac{\phi_W}{2K_W}\hat{\varrho}_\beta \psi_\beta, \quad \psi = \hat{s}^2, \quad \psi_\beta = \frac{\partial \psi}{\partial \varrho_\beta}$$

therefore...

effective stress

general structure

$$\hat{s}^2 = \psi(\varrho_A, \varrho_W, \varrho_S)$$

$$P_T = u_A - \chi(u_A - u_w)$$

$$\chi = \chi(\varrho_A, \varrho_W, \varrho_S) = \phi_W + \phi_s \left[\frac{\hat{\varrho}_A \psi_A - \hat{\varrho}_S \psi_S}{\hat{\varrho}_A \psi_A - \hat{\varrho}_W \psi_W} \right]$$

$$s = s(\varrho_A, \varrho_W, \varrho_S) = \frac{\phi_W}{2K_W} (\hat{\varrho}_A \psi_A - \hat{\varrho}_W \psi_W)$$

effective stress

SWRC dependent on saturation only

$$\hat{s}^2 = \psi(S_r)$$

$$P_T = u_A - \chi(u_A - u_w)$$

$$\chi = \chi(S_r) = S_r$$

$$s = s(S_r) = -\frac{S_r \psi_{sr}}{2K_W}$$

effective stress

SWRC dependent on saturation and porosity

$$\hat{s}^2 = \psi(S_r, n)$$

$$P_T = u_A - \chi(u_A - u_w)$$

$$\chi = \chi(S_r, n) = S_r - n(1-n)\frac{\psi_n}{\psi_{sr}}$$

$$s = s(S_r, n) = -\frac{S_r \psi_{sr}}{2K_W}$$

effective stress

example

$$\hat{s}^2 = \psi = 2AK_W\phi_S^\beta \left[\frac{1-\alpha+\alpha S_r - S_r^\alpha}{\alpha(1-\alpha)S_r^\alpha}\right]$$

$$P_T = u_A - \chi(u_A - u_w)$$

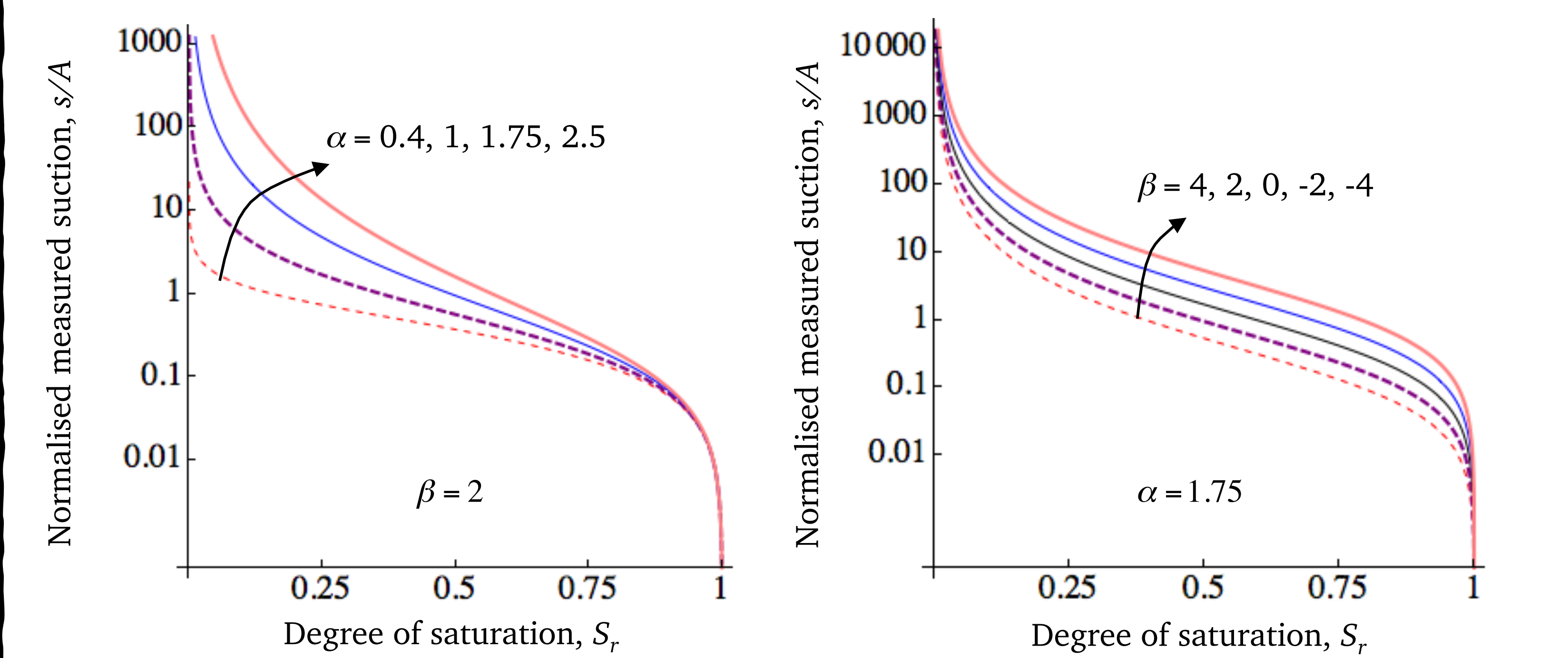
$$\chi = S_r - \beta S_r(1-n)\left[\frac{1-\alpha+\alpha S_r - S_r^\alpha}{\alpha(1-\alpha)(1-S_r)}\right]$$

$$s = A(1-n)^\beta\left[\frac{1-S_r}{S_r^\alpha}\right]$$

effective stress

example

$$s = A(1-n)^{\beta}\left[\frac{1-S_r}{S_r^{\alpha}}\right]$$

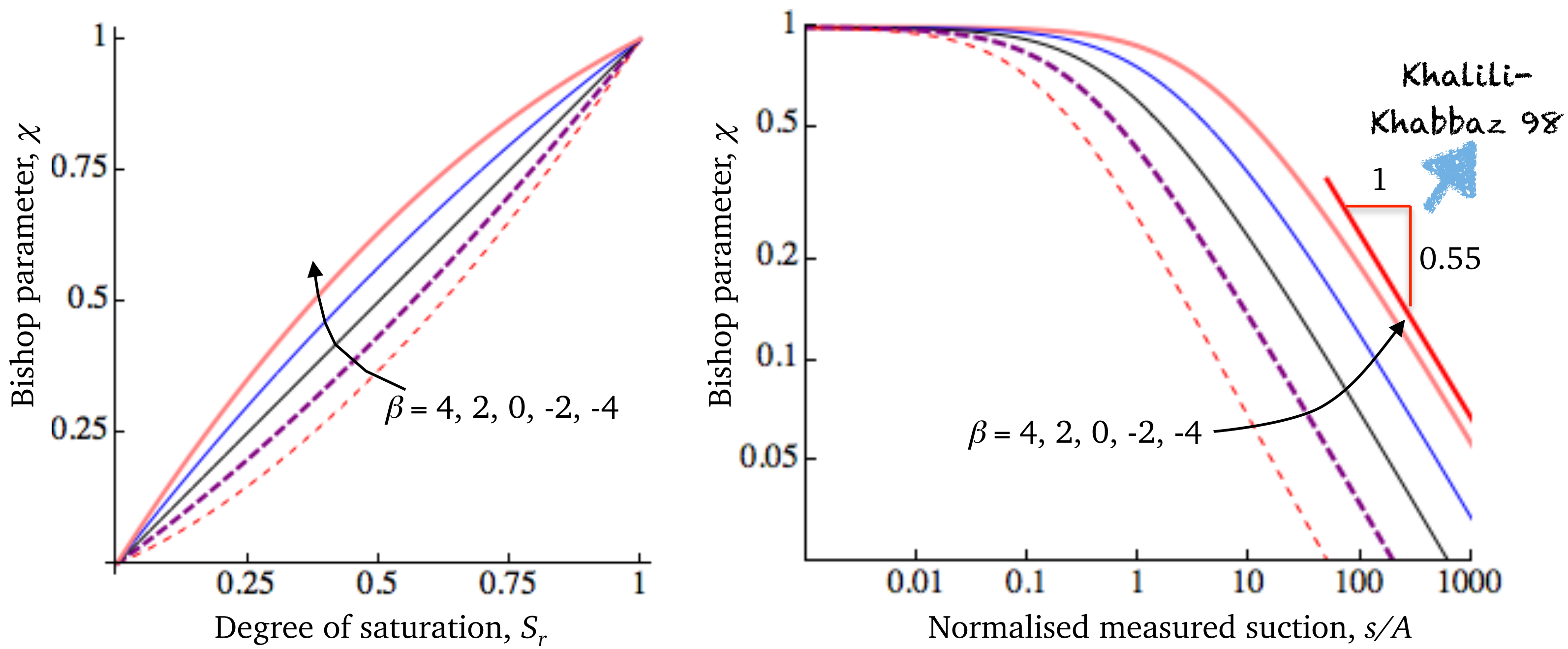


effective stress

example

$$s = A(1-n)^{\beta}\left[\frac{1-S_r}{S_r^{\alpha}}\right]$$

$$\alpha = 1.75, \quad n = 0.4$$

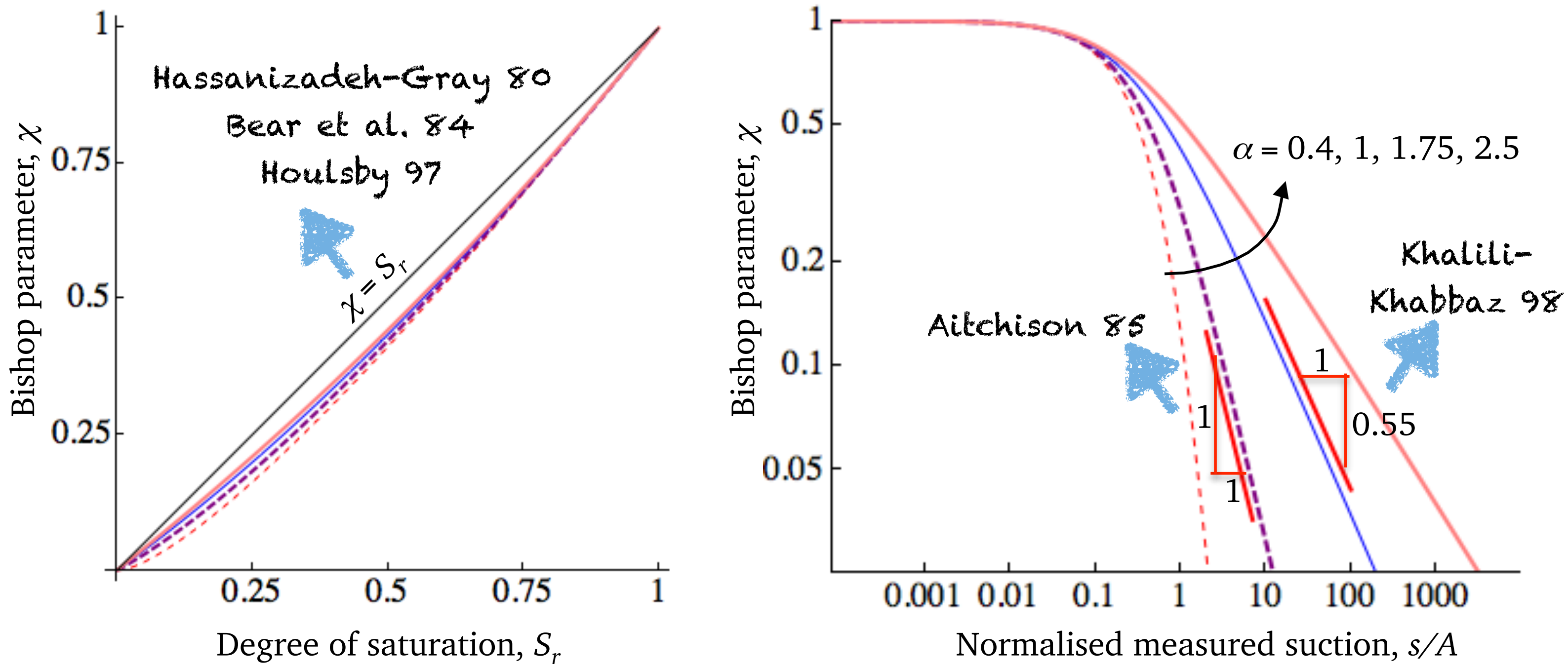


effective stress

example

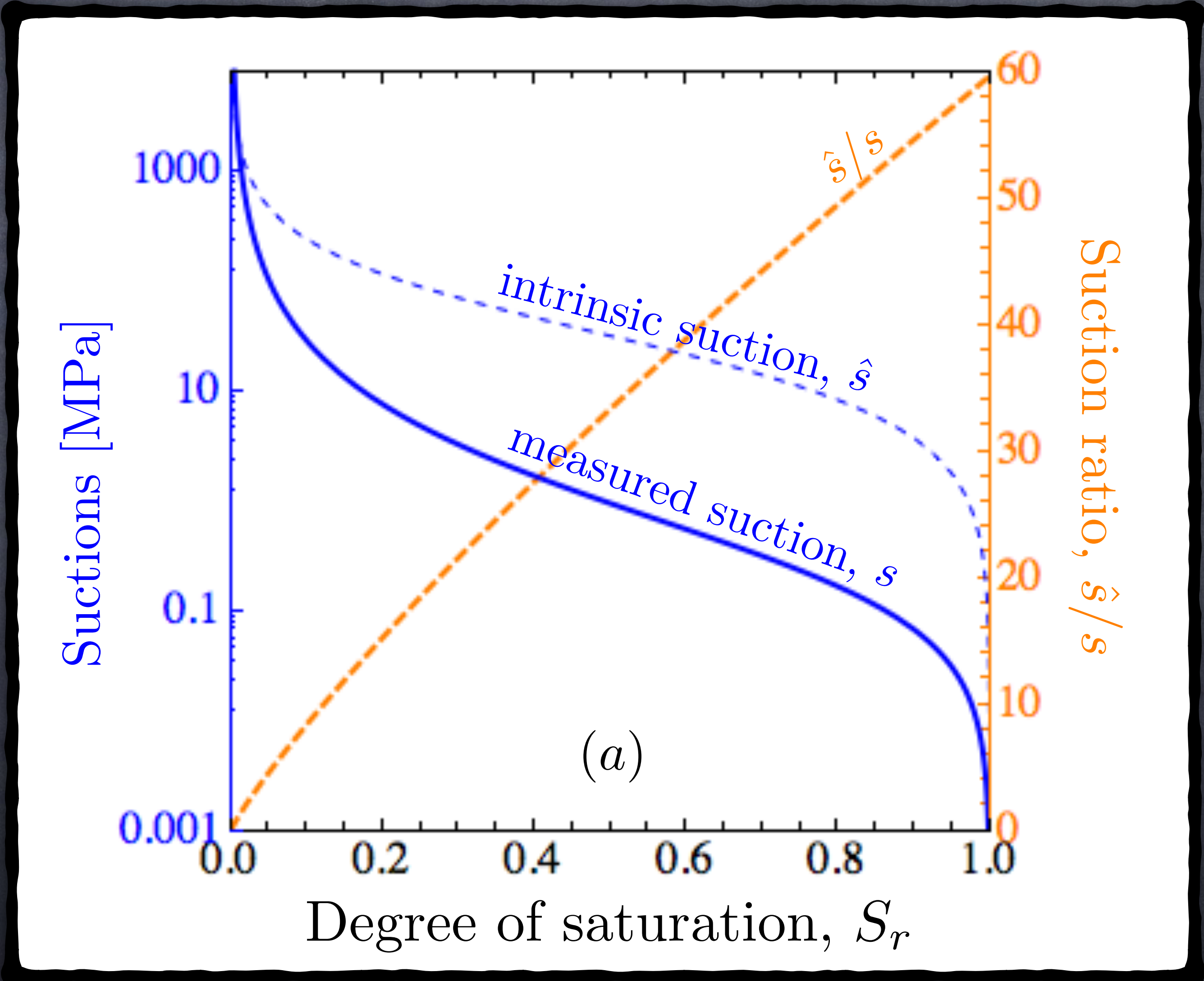
$$s = A(1-n)^{\beta}\left[\frac{1-S_r}{S_r^{\alpha}}\right]$$

$$\beta = 2, \quad n = 0.4$$



effective stress

example



conclusions

- hydrodynamics reveal the structure of effective stress in partially saturated soils
- its structure agrees with Bishop, but the χ parameter generally depends on 3 densities in a way strictly connected to soil-water retention
- realistic soil-water retention curves that are a function of both n and Sr, explain the versatility of observed experimental χ values.

what's not included
(left for future work)

- cohesion effects in very low saturations
- mass exchanges between phases